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Generalized Lyapunov Exponents in High-Dimensional Chaotic Dynamics and Products of Large Random Matrices

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We study the behavior of the generalized Lyapunov exponents for chaotic symplectic dynamical systems and products of random matrices in the limit of large dimensions D. For products of random matrices without any particular structure the generalized Lyapunov exponents become equal in this limit and the value of one of the generalized Lyapunov exponents is obtained by simple arguments. On the contrary, for random symplectic matrices with peculiar structures and for chaotic symplectic maps the generalized Lyapunov exponents remains different for $D \to \infty$, indicating that high dimensionality cannot always destroy intermittency.

KEY WORDS: Lyapunov exponents; generalized Lyapunov exponents; intermittency; products of random matrices; chaotic symplectic maps.

1. INTRODUCTION

Ergodicity is one of the most relevant properties for a statistical description of dynamical systems. One can see by numerical experiments,[(1)] and sometimes by analytical computations,[(2)] that chaoticity (in the sense that nearby trajectories diverge exponentially in time) is a feature common to a wide class of ergodic systems.

The degree of chaoticity is usually measured by the typical exponential growth of the uncertainty in the initial state of the system, i.e., by the

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maximal Lyapunov exponent λ. However, λ does not give a full description of the chaotic flow, since it is an asymptotic quantity. In general, one in fact observes variations of the degree of chaoticity on finite times. We call this phenomenon *temporal intermittency*; see Section 2 for technical details. The most impressive examples are given by one-dimensional maps with "quasitangent" contact and by the Lorentz system for certain values of the control parameters[3] where one has a regular motion for long times interrupted by randomly distributed bursts of strong chaoticity.

One can give a quantitative description of intermittency by introducing a set of *generalized* Lyapunov exponents $L(q)$ which describes the average growth of the moments of the response of the system to a perturbation.[4] This method gives a good description of the fluctuations of the chaoticity since it takes into account the finite-time properties of the flow. In the limit of nonintermittent systems it is easy to see that $L(q) = \lambda q$, but in general this is not true and the deviation of $L(q)$ from the linear behavior gives an indication of the intermittency degree of the flow.

Numerical computations show that intermittency occurs in generic systems with few degrees of freedom.[4] Therefore, it is natural to ask if intermittency disappears in the limit of infinitely many degrees of freedom (*thermodynamic limit*). This question is relevant in turbulence[5] and in explosion problems,[6] where stochastic linear differential equations with multiplicative noise are involved (see Appendix A).

In order to investigate this point, we have studied products of random matrices and symplectic maps in the limit of large dimension D. Let us briefly state our results:

1. For products of random matrices without particular structure (e.g., with independent, identically distributed elements) the intermittency disappears for $D \to \infty$, i.e., $L_D(q) = \lambda(D) q + O(1/D^\eta)$, where η depends on the details of the probability distribution of the elements. Moreover, we find that $\lambda(D) = \lambda(\infty) + O(1/D^\eta)$, where $\lambda(\infty)$ can be obtained by a simple argument.

2. In the case of products of particular symplectic random matrices with high connectance (i.e., with a number of random elements $\propto D^2$) we obtain results similar to those of the previous point, but now the asymptotic value $\lambda(\infty)$ cannot be obtained by trivial arguments.

3. For the same type of symplectic random matrices but with low connectance as well as for symplectic maps we find that intermittency does not disappear in the $D \to \infty$ limit, i.e., $L_\infty(q) \neq \lambda(\infty) q$.

In Section 2 we briefly introduce the generalized Lyapunov exponents and discuss their meaning. We present the results for the products of random matrices without particular structure in Section 3 and for sympletic maps and products of symplectic random matrices in Section 4. Section 5 is devoted to the conclusions. In Appendix A we report the calculation of $L(q)$ for stochastic linear differential equations with multiplicative noise and Appendix B the computation for products of random matrices with independent elements whose mean value is different from zero.

2. GENERALIZED LYAPUNOV EXPONENTS

Let us consider a deterministic map

$$\mathbf{x}(n+1) = \mathbf{G}[\mathbf{x}(n)], \qquad \mathbf{x}, \mathbf{G} \in \mathbf{R}^D, \qquad n = 0, 1, \ldots \tag{2.1}$$

The maximal Lyapunov exponent λ is defined[7] considering the linear evolution of the tangen vector $\zeta \in \mathbf{R}^D$:

$$\zeta_i(n+1) = \sum_{j=1}^{D} \frac{\partial G_i[\mathbf{x}(n)]}{\partial x_j} \zeta_j(n) \tag{2.2}$$

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \ln \frac{|\zeta(n)|}{|\zeta(0)|} \tag{2.3}$$

It follows that a small uncertainty $\delta \mathbf{x}(0)$ in the initial state of the system becomes, after n iterations,

$$|\delta \mathbf{x}(n)| \sim |\delta \mathbf{x}(0)| \, e^{\lambda n} \tag{2.4}$$

For a product of $D \times D$ random matrices $\prod_{k=1}^{n} \mathsf{A}(k) = \mathsf{A}(n)\, \mathsf{A}(n-1) \cdots \mathsf{A}(1)$ the definition of λ is still given by Eq. (2.3), where now

$$\zeta(n) = \mathsf{A}(n)\, \zeta(n-1) \tag{2.5}$$

The meaning of λ in this case depends on the particular problem connected to (2.5). For example, λ is the inverse of the characteristic localization length in a one-dimensional discrete Schrödinger equation with random potential[8] and is proportional to the free energy in one-dimensional disordered systems.[9] The exponent λ, however, does not describe the degree of intermittency, because of its global character.

It is useful to define the response $R_m(n)$ to a perturbation in $\mathbf{x}(m)$ after a time n by

$$R_m(n) = \frac{|\zeta(m+n)|}{|\zeta(m)|}$$

so that λ is given by

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \langle \ln R(n) \rangle \tag{2.6}$$

where $\langle \cdots \rangle$ indicates a time average, i.e.,

$$\langle f(\mathbf{x}) \rangle = \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} f(\mathbf{x}(k))$$

The usual definition of λ does not require an average, since $\lambda = \lim_{n \to \infty}(1/n) \ln R_m(n)$ for almost all initial conditions m. As consequence of the Oseledec theorem,[10] the two definitions in fact are equivalent.

The generalized Lyapunov exponents are then defined as[4]

$$L(q) = \lim_{n \to \infty} \frac{1}{n} \ln \langle R^q(n) \rangle \tag{2.7}$$

One can easily recognize that

$$\lambda = \left. \frac{dL(q)}{dq} \right|_{q=0} \tag{2.8}$$

A linear behavior $L(q) = \lambda q$ indicates absence of intermittency; in general, however, $L(q)$ is a convex funtion of q.[11]

We want briefly to stress that basically $L(q)$ gives an indication of the large fluctuations of $R(n)$ at finite n.[12] Let us define a local exponent parameter (LEP) γ as

$$R_m(n) \sim e^{\gamma(m)n}, \qquad n \gg 1 \tag{2.9}$$

and classify the trajectories of length n (i.e., $\{\mathbf{x}(m), \mathbf{x}(m+1),..., \mathbf{x}(m+n)\}$) according their LEP.

In order to obtain an exponential growth of $\langle R^q(n) \rangle$, we have to assume that the probability of having a response, after n steps, with a given LEP has the form

$$dP_n(\gamma) = d\mu(\gamma) e^{-S(\gamma)n}, \qquad S(\gamma) \geqslant 0 \tag{2.10}$$

where $\mu(\gamma)$ is a smooth function of γ. We can now calculate $\langle R^q(n) \rangle$ averaging over the γ distribution, i.e.,

$$\langle R^q(n) \rangle = \int d\mu(\gamma)\, e^{[q\gamma - S(\gamma)]n} \sim e^{nL(q)} \tag{2.11}$$

For large n the integral can be calculated by the saddle point method, and we find

$$L(q) = \max[q\gamma - S(\gamma)] \tag{2.12}$$

The Legendre transform (2.12) shows that each value of q selects a particular $\bar{\gamma}$ given by

$$q = \left.\frac{dS(\gamma)}{d\gamma}\right|_{\bar{\gamma}}$$

We note that the function S has its minimum value ($S = 0$) at $\gamma = \lambda$ as consequence of the Oseledec theorem.[10] It is also possible to see that this description in terms of γ and $S(\gamma)$ is related to a multifractal structure in a suitable space. We are not interested here in this aspect of the problem; for a review see ref. 13.

The probability distribution of $R(n)$ for $n \gg 1$ is usually close to a log-normal distribution:

$$P[R(n)] \simeq \frac{1}{R(n)(2\pi\mu n)^{1/2}} \exp\left[\frac{-[\ln R(n) - \lambda n]^2}{2\mu n}\right] \tag{2.13}$$

where

$$\mu = \lim_{n \to \infty} \frac{1}{n} \langle [\ln R(n) - \lambda n]^2 \rangle$$

Note indeed that writing $n = \tilde{n}\Delta$ (with $\tilde{n}$ and Δ large), one has $R_m(n) = \prod_{k=1}^{\tilde{n}} \tilde{R}(k)$ with $\tilde{R}(k) = R_{m+(k-1)\Delta}$. Therefore, since $\ln \tilde{R}(k)$ are practically uncorrelated variables, we can use the central limit theorem for $\ln R_m(n)$ and, after a change of variables, we obtain Eq. (2.13). Under the hypothesis that $R(n)$ is exactly a log-normal variable, one has

$$L(q) = \lambda q + \tfrac{1}{2}\mu q^2 \tag{2.14}$$

In general, Eq. (2.14) is true only for small q, even if the log-normal is a good approximation. This trouble is due to the fact that the moments of the log-normal distribution grow very fast with q.[14]

At the first rough level we have two relevant parameters for the characterization of intermittency: λ and μ. One can show that the value $\mu/\lambda = 1$ delimits the borderline between weak and strong intermittency. Since $P[R(n)]$ reaches its maximum for

$$R^*(n) = e^{\lambda n(1 - \mu/\lambda)}$$

one sees that in the case $\mu/\lambda > 1$ intermittency gives drastic corrections to the "mean field" result obtained taking into account only the maximum of the probability distribution. In fact, for $n \to \infty$, $R^*(n) \to 0$, while $\langle \ln R(n) \rangle \to \infty$.

3. PRODUCTS OF RANDOM MATRICES WITHOUT STRUCTURES

In this section we consider products $\prod_{k=1}^{n} \mathsf{A}(k)$ of $D \times D$ independent random matrices $\mathsf{A}(k)$ with no particular structure. We mean that the matrix $\mathsf{A}(k)$ is independent of the matrix $\mathsf{A}(h)$ if $k \neq h$, and that each element $\mathsf{A}(k)_{ij}$ is independent of $\mathsf{A}(k)_{rs}$ if $(i, j) \neq (r, s)$. We here take into account the symmetry as the only possible "structure" [i.e., $\mathbf{A}(k)_{ij} = \mathbf{A}(k)_{ji}$].

We consider two cases: (1) $\overline{\mathbf{A}(k)_{ij}} = 0$, (2) $\overline{\mathbf{A}(k)_{ij}} > 0$, where the $\overline{(\cdots)}$ indicates the average over the probability distribution of the matrix elements. In the first case we have chosen the normalization

$$\mathsf{A}(k)_{ij} = \frac{x(k)_{ij}}{\sqrt{D}} \tag{3.1}$$

and in the second case

$$\mathsf{A}(k)_{ij} = \frac{\bar{y}}{D} + \frac{x(k)_{ij}}{D} \tag{3.2}$$

where $x(k)_{ij}$ are identically distributed independent random variables with zero mean and variance $\sigma^2 = \overline{x(k)_{ij}^2}$ and $\bar{y} > 0$ is a fixed number. The scalings (3.1) and (3.2) are the natural ones in order to have $L(q) \sim O(1)$ in the thermodynamic limit $D \to \infty$.

For products of random matrices the exponents $L(q)$ can be defined as[15]

$$L(q) = \lim_{n \to \infty} \frac{1}{n} \ln \left\langle \left| \operatorname{Tr} \prod_{k=1}^{n} \mathsf{A}(k) \right|^{q} \right\rangle \tag{3.3}$$

which is equivalent to (2.7). Using the independence of $\mathsf{A}(k)_{ij}$ in (3.3), one sees that in the case (3.1)

$$L(2) = \ln \sigma^2 \tag{3.4a}$$

while in the case (3.2)

$$L(1) = \ln \bar{y} \tag{3.4b}$$

at least for large D.

As far as we know, there is only one analytical result for matrices of the form (3.1) with $x(k)_{ij}$ standard symmetric stable random variables of exponent α.[16] In the case $\alpha = 2$ (i.e., Gaussian variables) one has

$$\lambda(D) = \frac{1}{2}\left(\ln \sigma^2 - \frac{1}{D}\right) + O\left(\frac{1}{D^2}\right) \tag{3.5}$$

and therefore

$$\mu(D) = O\left(\frac{1}{D}\right) \tag{3.6}$$

Note that $\lambda(D) \to L(2)/2$ and $\mu(D) \to 0$ as $D \to \infty$; in the limit of large dimensions the intermittency therefore disappears and $L(2)$ is obtained by a trivial argument. The behavior $\lambda(\infty) - \lambda(D) \propto 1/D$ is shown in Fig. 1 as

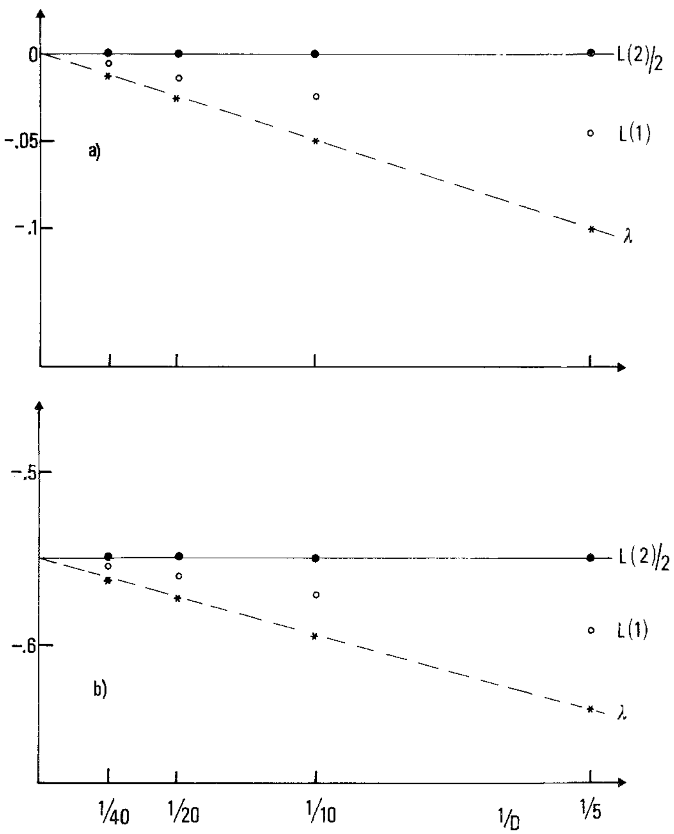

Fig. 1. λ, $L(1)$, and $L(2)/2$ vs. $1/D$ for matrices of the form (3.1): (a) $\sigma = 1$, $x(k)_{ij}$ are Gaussian variables, and $x(k)_{ij} = x(k)_{ji}$; the dashed line indicates Eq. (3.5). (b) $x(k)_{ij}$ are random variables uniformly distributed in the interval $[-1, 1]$; the horizontal line indicates Eq. (3.4a).

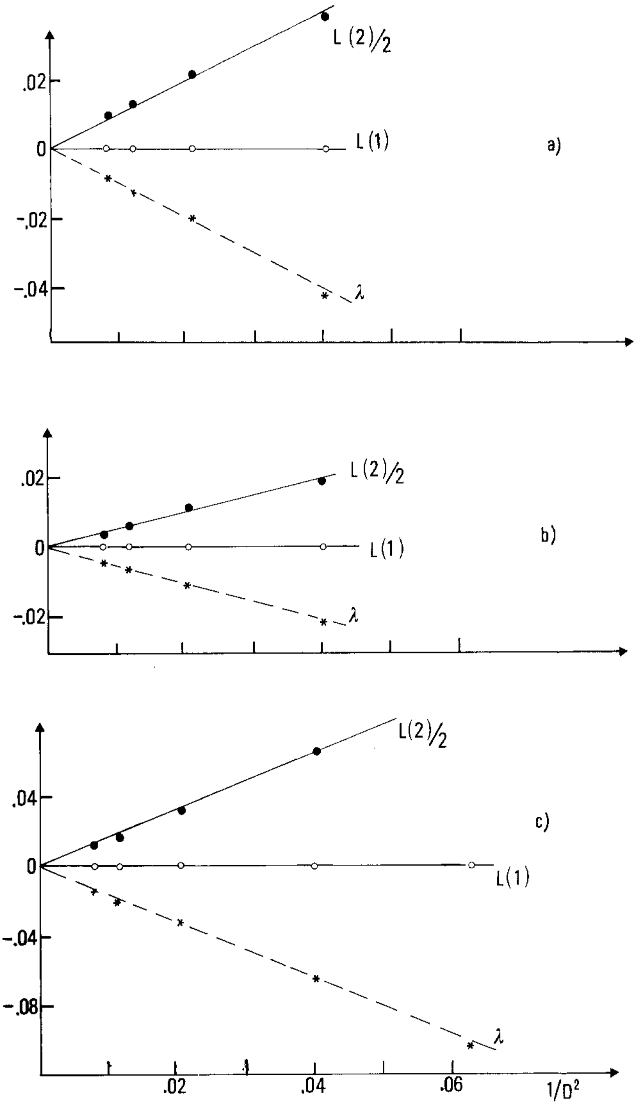

Fig. 2. λ, $L(1)$, and $L(2)/2$ vs. $1/D^2$ for matrices of the form (3.2). The horizontal line indicates Eq. (3.4b) and the dashed line indicates Eq. (3.8). (a) $\bar{y}=1$, $\sigma=1$; $x(k)_{ij}$ are Gaussian variables, and the matrix $\mathsf{A}(k)$ is symmetric; (b) As in (a), but the matrix $\mathsf{A}(k)$ is not symmetric; (c) $x(k)_{ij}$ are random variables uniformly distributed in the interval $[-1, 1]$.

observed in numerical calculations of products of matrices with the form (3.1), independent of the details of the probability distribution of the $x(k)_{ij}$ even in the case of symmetric matrices.

Let us now consider matrices of the form (3.2). In this case we are able to prove, by means of a perturbative calculation and neglecting terms of $o[(\sigma/D)^2]$ (see Appendix B), that

$$L_D(2) = 2 \ln \bar{y} + c \left(\frac{\sigma}{D\bar{y}}\right)^2 \tag{3.7}$$

and since Eq. (2.14) for $q = 2$ is valid when D is large enough (as numerically checked), one gets

$$\lambda(D) = \ln \bar{y} - \frac{c}{2}\left(\frac{\sigma}{D\bar{y}}\right)^2 \tag{3.8}$$

where $c = 2$ if we impose the $A(k)$ to be symmetric, and $c = 1$ if we do not. Figure 2 shows a comparison between the numerical results and Eq. (3.8). Also in this case intermittency disappears in the large-D limit and $\lambda(\infty) = L(1)$.

We stress that, in order to have $L(2) < \infty$, in our calculations we always consider distributions of $\mathrm{A}(k)_{ij}$ with finite variance. In ref. 16 this condition is satisfied only for $\alpha = 2$. The situations with $\sigma^2 = \infty$ [i.e., $L(2) = \infty$] should give a rather different scenario for $L(q)$. We expect, in fact, a finite value of $L(q)$ only for $q < q_c < 2$, as found in a particular case.[17]

4. SYMPLECTIC MAPS AND PRODUCTS OF RANDOM SYMPLECTIC MATRICES WITH STRUCTURES

Let us consider symplectic maps of the form

$$\begin{aligned} \mathbf{q}(k+1) &= \mathbf{q}(k) + \mathbf{p}(k), \qquad \mathrm{mod}(2\pi) \\ \mathbf{p}(k+1) &= \mathbf{p}(k) - \varepsilon \nabla F[\mathbf{q}(k+1)] \end{aligned} \tag{4.1}$$

where $\mathbf{q}, \mathbf{p} \in \mathbf{R}^D$ and $\nabla = (\partial/\partial q_1, \ldots, \partial/\partial q_D)$. We point out that a symplectic map is a canonical transformation from the variables $(\mathbf{q}(k), \mathbf{p}(k))$ to $(\mathbf{q}(k+1), \mathbf{p}(k+1))$. Moreover, such a map can be interpreted as the recursive rule related to a Poincaré section of a Hamiltonian system with $D+1$ degrees of freedom. Note that for $\varepsilon = 0$ the map (4.1) represents a system of D uncoupled ocillators ($\mathbf{p}$ being the actions and $\mathbf{q}$ the angles). For $\varepsilon \neq 0$, εF plays the role of the nonintegrable part in the Hamiltonian flux.

We consider as first case nearest neighbor interactions and periodic boundary conditions, i.e.,

$$F=\sum_{i=1}^{D} f[q_{i+1}-q_i]$$

and

$$\begin{aligned}
q_i(k+1) &= q_i(k)+p_i(k), \qquad \mathrm{mod}(2\pi)\\
p_i(k+1) &= p_i(k)+\varepsilon\{g[q_{i+1}(k+1)-q_i(k+1)]\\
&\quad -g[q_i(k+1)-q_{i-1}(k+1)]\}
\end{aligned}$$

with $q_1=q_{D+1}$, $p_1=p_{D+1}$, and $g(x)=df(x)/dx$. In our simulations we have chosen g of the form

$$g(x)=\sin^{\beta}(x)$$

where β is an odd integer. In Fig. 3, λ, $L(1)$, and $L(2)/2$ are plotted as a function of $1/D$ for different values of ε and β. It is well evident that in all these cases $\mu(\infty)\neq 0$. We want to note that the intermittency may be relevant in the sense that $\mu/\lambda>1$ even when $D\to\infty$; see Fig. 3c.

Nevertheless, one could suspect that the persistence of intermittency up to the thermodynamic limit is pathologically related to our particular chaotic systems. A way to investigate this point consists in neglecting the deterministic correlations in the evolution of the tangent vectors. The computation of $L(q)$ for the symplectic map (4.1) involves products of matrices of the form

$$\mathsf{B}(k)=\begin{pmatrix} \mathsf{1} & \mathsf{1} \\ \varepsilon\mathsf{b}(k) & \mathsf{1}+\varepsilon\mathsf{b}(k)\end{pmatrix} \tag{4.2}$$

where $\mathsf{1}$ is the $D\times D$ identity matrix and $\mathsf{b}(k)$ is the $D\times D$ symmetric matrix defined by

$$\mathsf{b}(k)_{ij}=\frac{\partial^2 F[\mathbf{q}(k)]}{\partial q_i\,\partial q_j} \tag{4.3}$$

where $\mathbf{q}(k)$ is given by the map (4.1). In the highly chaotic regime, this deterministic dynamics can be approximated, in a nontrivial way, by a product of symplectic random matrices $\mathsf{A}(k)$ of the form[18]

$$\mathsf{A}(k)=\begin{pmatrix} \mathsf{1} & \mathsf{1} \\ \varepsilon\mathsf{a}(k) & \mathsf{1}+\varepsilon\mathsf{a}(k)\end{pmatrix} \tag{4.4}$$

where a(k) is a symmetric random matrix. Essentially the idea is that the randomness of A(k) mimics the chaoticity of the trajectories generated by the deterministic dynamics given by the map (4.1). It is interesting to note that the Lyapunov exponent for products of random matrices of the form (4.4) shows behavior (often not only qualitatively[19]) very close to those obtained with the *true* dynamics (4.1). Beyond the relation with deterministic chaotic systems, products of random symplectic matrices are interesting in themselves. For a recent review on random matrices see, e.g., ref. 20.

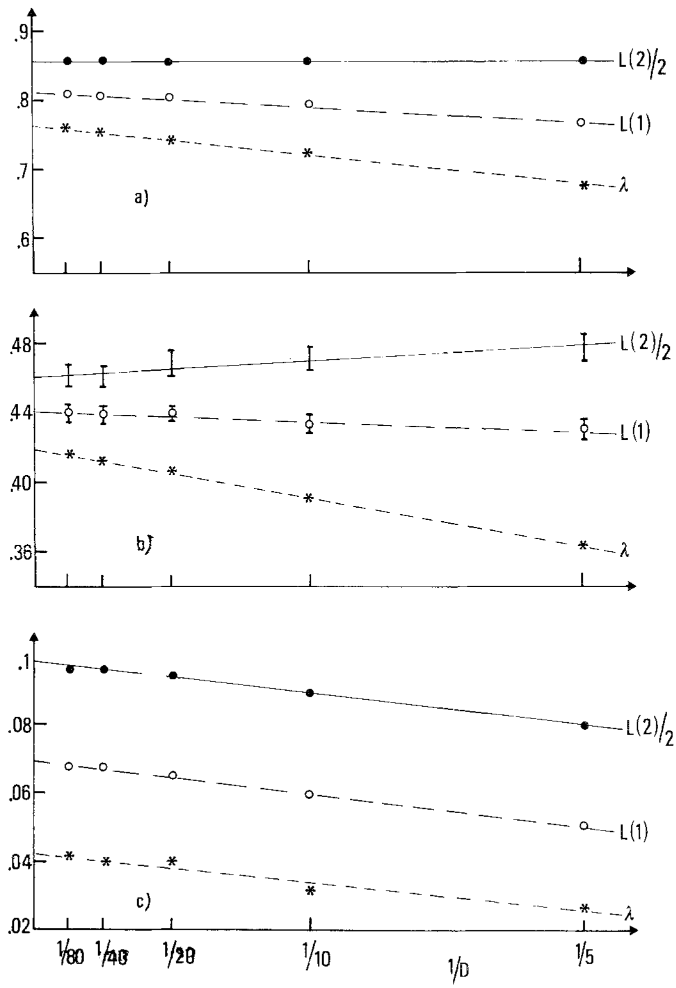

Fig. 3. λ, $L(1)$, and $L(2)/2$ vs. $1/D$ for symplectic maps (4.1): (a) $\beta=1$, $\varepsilon=1$; (b) $\beta=3$, $\varepsilon=0.4$; (c) $\beta=5$, $\varepsilon=0.02$.

In order to represent nearest neighbor coupling, we assume that the elements $\mathsf{a}(k)_{ij}$ are nonzero only if $|i-j| \leqslant 1$ or $(i, j) = (1, D)$, $(D, 1)$. In the case $\overline{\mathsf{a}(k)_{ij}} = 0$ it has been shown,[20] by means of a perturbative expansion, that

$$L(2) = (2z\overline{a^2})^{1/3}\, \varepsilon^{2/3} \tag{4.5}$$

where z is the number of nonzero elements on each line of $\mathsf{a}(k)$ (in the nearest neighbor case $z = 3$), and $\overline{a^2}$ is the variance of the distribution of the nonzero elements of $\mathsf{a}(k)$.

In Fig. 4 the behavior of λ, $L(1)$, and $L(2)/2$ is shown as a function of $1/D$ for different values of ε in the case when the nonzero elements of $\mathsf{a}(k)$

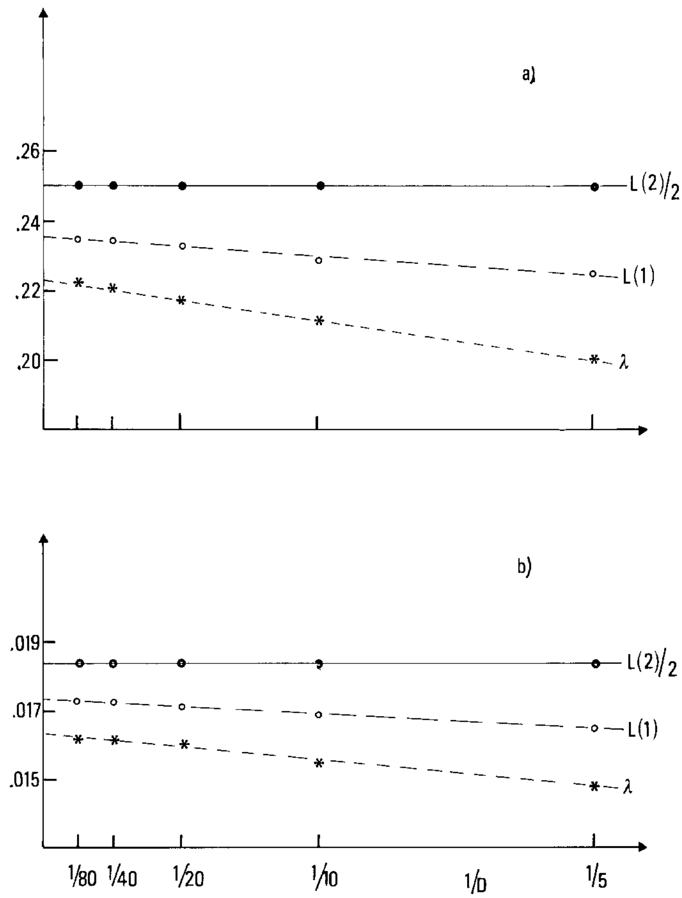

Fig. 4. λ, $L(1)$, and $L(2)/2$ vs. $1/D$ for symplectic random matrices (4.4) with $z = 3$; the horizontal lines indicates Eq. (4.5). (a) $\varepsilon = 0.5$, (b) $\varepsilon = 0.01$.

are uniformly distributed in the interval $[-1/2, 1/2]$. The behavior is similar to that observed in Fig. 3 for the map (4.1). In the matrices considered in this section the number of nonzero random elements is $O(D)$ while the total number of elements is $4D^2$. Therefore, we are in a situation of *low connectance*, i.e., of finite-range interactions.

In order to investigate the role of the connectance in the intermittency, we have performed numerical calculations for different values of z. In Fig. 5 we plot λ and $L(2)/2$ as function of $1/D$ for $z=D$, i.e., all the elements of $\mathsf{a}(k)$ are nonzero. To keep $L(2)$ fixed, we have rescaled ε by a factor $D^{-1/2}$. In this case of maximal connectance, we observe that $\lambda(D) \to L(2)/2$ and $\mu(D) \propto 1/D$ for large D. This behavior remains valid for high connectance where $z \propto D$ and the number of nonzero random elements of $\mathsf{a}(k)$ is $O(D^2)$. On the contrary, for low connectance, i.e., z independent of D, one has the same behavior as for the case $z=3$.

Symplectic maps exhibit the same scenario which has been described for a product of symplectic random matrices of the form (4.4), considering in (4.1) F as follows:

$$F = \sum_{i=1}^{D} \sum_{j=1}^{\tilde{n}} f[q_{i+j} - q_i]$$

where $\tilde{n}$ plays the role of z.

It is worth stressing that the high-connectance situation is intermediate between the case of random matrices without structure, discussed in the

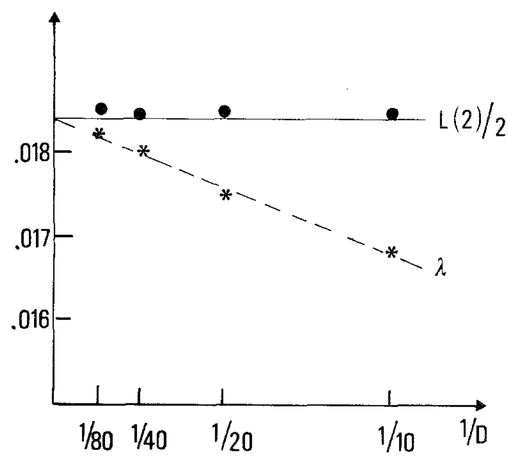

Fig. 5. λ and $L(2)/2$ vs. $1/D$ for symplectic random matrices (4.4) with $z=D$ and $\varepsilon=0.01/(D/3)^{1/2}$; the horizontal line indicates Eq. (4.5).

previous section, and the case of symplectic maps and random matrices with low connectance. Indeed, even if we have that $\lambda(D) \to L(2)/2$ when $D \to \infty$, the value of $L(2)$ cannot be obtained by trivial arguments.

5. CONCLUSIONS

We have studied the problem of the generalized Lyapunov exponents in the high-dimensional limit for the following systems: (1) products of random matrices without structure; (2) products of random symplectic matrices; (3) chaotic symplectic maps.

The main result is that in the cases of high connectance the intermittency disappears when the dimension of the system is increased, while for systems with low connectance (i.e., finite-neighbor interactions) the intermittency survives also when $D \to \infty$. As a consequence, for these systems the finite-time fluctuations play a relevant role in this thermodynamic limit. These are indeed the most generic cases. Finally, we have extended our results to linear stochastic differential equations with multiplicative noise.

APPENDIX A. GENERALIZED LYAPUNOV EXPONENTS FOR LINEAR STOCHASTIC DIFFERENTIAL EQUATIONS WITH MULTIPLICATIVE NOISE

Let us consider the following linear stochastic differential equation with multiplicative noise (we adopt the Stratanovich calculus[22]):

$$dx_i = \sum_{j=1}^{D} \mathsf{A}_{ij} x_j \, dt + \sum_{j=1}^{D} x_j \, d\mathsf{W}_{ij} \tag{A.1}$$

where W_{ij} are Wiener processes, i.e., uncorrelated Gaussian processes with zero mean and

$$\langle\!\langle d\mathsf{W}_{ij} \, d\mathsf{W}_{kl} \rangle\!\rangle = \frac{\sigma^2}{D} \delta_{ik} \delta_{jl} \, dt \tag{A.2}$$

where $\sigma = O(1)$ and A is $D \times D$ (random quenched) matrix whose elements are constant in time of order $O(D^{-1/2})$ of both signs and $\langle\!\langle \cdots \rangle\!\rangle$ denotes the average over the Wiener measure. Note that this normalization is quite natural in order to have $dx_i = a_i(\mathbf{x}) \, dt + b_i(\mathbf{x}, \tilde{\mathbf{w}}) \, d\tilde{w}_i$ with a_i and b_i of order $O(1)$ and $\tilde{w}_i$ a standard Wiener process. Equation (A.1) models some explosion problems with random control parameters.[6] Moreover, it is involved in the problem of the stretching of a line in a D-dimensional random straining velocity field.[5]

Let us in fact consider an infinitesimal line element $\mathbf{x} = (x_1,..., x_D)$ moving with the field velocity $\mathbf{v}(\mathbf{y}, t)(\mathbf{v}, \mathbf{y} \in \mathbf{R}^D)$. This is strecthed according to

$$dx_i = \sum_{j=1}^{D} \frac{\partial v_i}{\partial y_j} x_j \, dt \tag{A.3}$$

Thus, Eq. (A.1) is obtained assuming that $\partial v_i / \partial y_j$ consists of two parts, one steady and the other given by a quickly varying term:

$$\frac{\partial v_i}{\partial y_j} dt = \mathsf{A}_{ij} \, dt + d\mathsf{W}_{ij}$$

We show now that for the stochastic process defined by Eq. (A.1) one has

$$L(q) = \lambda q + \frac{a}{D} q^2 \tag{A.4}$$

where

$$\lambda = \max_i \operatorname{Re} l_i = O(1) \tag{A.5}$$

$\{l_i\}$ are the eigenvalues of the matrix A, $a = O(1)$, and Re indicates the real part. If A is a random matrix, then Eq. (A.5) follows directly from the generalized circular law for the average distribution of the eigenvalues of random matrices.[23]

With a suitable change of variables $\mathbf{z} = \mathsf{U}\mathbf{x}$, Eq. (A.1) can be written as

$$dz_i = l_i z_i \, dt + \sum_{j=i}^{D} z_j \, d\tilde{\mathsf{W}}_{ij} \tag{A.6}$$

where for simplicity of notation we have assumed that the eigenvalues are not degenerate, and $d\tilde{\mathsf{W}}_{ij} = \sum_{kl} \mathsf{U}_{ik} \, d\mathsf{W}_{kl} \, \mathsf{U}_{lj}^{-1}$. Note that since $\mathsf{U}_{ij} = O(D^{-1/2})$ [we remember that $\mathbf{UAU}^{-1}$ is diagonal with elements $O(1)$] we have that $\tilde{\mathsf{W}}_{ij}$ are Wiener processes, so that from the law of large number, for $D \gg 1$ one has

$$\langle\langle d\tilde{\mathsf{W}}_{ij} \, d\tilde{\mathsf{W}}_{kl} \rangle\rangle = \begin{cases} \dfrac{c_{ij}}{D} dt, & c_{ij} = O(1) \quad \text{if} \quad (i, j) = (k, l) \\[2ex] \dfrac{c_{ijkl}}{D^2} dt, & c_{ijkl} = O(1) \quad \text{if} \quad (i, j) \neq (k, l) \end{cases}$$

Let us now write down the evolution equation for $R^2 = \sum_{i=1}^{D} z_i^2$:

$$dR^2 = 2 \sum_{i=1}^{D} l_i z_i^2 \, dt + 2 \sum_{i,j} z_i z_j \, d\tilde{W}_{ij} \tag{A.7}$$

Defining

$$\begin{aligned} \sum_{i=1}^{D} l_i z_i^2 &= C_1(t) R^2 \\ \sum_{i,j}^{1,D} z_i z_j \, d\tilde{W}_{ij} &= \frac{R^2}{\sqrt{D}} C_2(t) \, d\tilde{w} \end{aligned} \tag{A.8}$$

with $C_1(t)$ and $C_2(t)$ bounded random variables, $C_1 \leqslant \max_i \operatorname{Re} l_i$, $C_2 \leqslant \text{const} = O(1)$, and $\tilde{w}$ a standard Wiener process, Eq. (A.7) becomes

$$dR^2 = 2C_1(t) R^2 \, dt + \frac{1}{\sqrt{D}} C_2(t) R^2 \, d\tilde{w} \tag{A.9}$$

Invoking ergodicity, one can compute the mximal Lyapunov exponent by an average over the Wiener measure:

$$\lambda = \lim_{t \to \infty} \frac{1}{2t} \langle\!\langle \ln R^2 \rangle\!\rangle = \langle\!\langle C_1 \rangle\!\rangle \leqslant \max_i \operatorname{Re} l_i \tag{A.10}$$

Denoting by the index for which $\operatorname{Re} l_i$ assumes its mximum value, we have

$$\lim_{t \to \infty} \frac{1}{t} \langle\!\langle \ln R^2 \rangle\!\rangle \geqslant \lim_{t \to \infty} \frac{1}{t} \langle\!\langle \ln z^2 \rangle\!\rangle = 2\lambda \tag{A.11}$$

Comparing Eqs. (A.10) and (A.11), Eq. (A.5) follows. Moreover, from Eq. (A.9) one has

$$\lim_{t \to \infty} \frac{1}{t} \langle\!\langle (\ln R - \langle \ln R \rangle)^2 \rangle\!\rangle = \frac{\text{const}}{D} \tag{A.12}$$

from which Eq. (A.4) follows. Therefore, we see that in the limit $D \to \infty$ the intermitency disappears in a way similar to that observed for products of random matrices without structure.

APPENDIX B. COMPUTATION OF THE GENERALIZED LYAPUNOV EXPONENTS FOR PRODUCTS OF RANDOM MATRICES WITHOUT STRUCTURE

We limit ourselves to the case $\overline{\mathsf{A}_{ij}} > 0$. Let us write the matrices of the form (3.2) in the following way:

$$\mathsf{A}(k) = \frac{\bar{y}}{D} \mathsf{a} + \frac{\sigma}{D} \mathsf{b}(k) \tag{B.1}$$

where $a_{ij} = 1$ and $b(k)_{ij}$ are independent random variables with zero mean and variance one. Since $L(1) = \ln \bar{y}$, in order to calculate λ, it suffices to evaluate $L(2)$, assuming that Eq. (2.14) holds for $q = 2$ (at least for large D).

By definition, $L(2)$ is given in the large-n limit by

$$\langle \zeta(n)^2 \rangle = \langle \zeta(0), B^T(n)\, B(n)\, \zeta(0) \rangle \sim e^{nL(2)} \tag{B.2}$$

where

$$B(n) = A(n)\, A(n-1) \cdots A(1) \tag{B.3}$$

We will follow the method of ref. 21 for the computation of $L(2)$ for products of matrices of the form (4.4). We write $B(n)$ as follows:

$$\begin{aligned} B(n) = \sum_{m=0}^{n} \left(\frac{\bar{y}}{D}\right)^{n-m} \left(\frac{\sigma}{D}\right)^{m} \\ \times \sum_{i_1 + \cdots + i_{m+1} = n-m} a^{i_1} b(i_{m+1} + \cdots + i_2 + m) a^{i_2} \cdots b(i_{m+1} + 1) a^{i_{m+1}} \end{aligned} \tag{B.4}$$

We need to estimate $B^T(n)\, B(n)$. Since $\overline{b_{ij}} = 0$ and since $b(k)$ is independent of $b(h)$ if $k \neq h$, the only contributions to the average (B.2) from each element of $B^T(n)\, B(n)$ are given by

$$\begin{aligned} &\sum_{m=0}^{n} \left(\frac{\bar{y}}{D}\right)^{2(n-m)} \left(\frac{\sigma}{D}\right)^{2m} \\ &\quad \times \sum_{i_1 + \cdots + i_m + 1 = n-m} (a^T)^{i_{m+1}} b^T(i_{m+1} + 1) \cdots \\ &\quad \times (a^T)^{i_1} b^T(i_{m+1} + \cdots + i_2 + m)(a^T)^{i_1} \\ &\quad \times a^{i_b} b(i_{m+1} + \cdots + i_2 + m) a^{i_2} \cdots b(i_{m+1} + 1) a^{i_m + 1} \end{aligned} \tag{B.5}$$

Noting that $a^j = D^{j-1} a$ and $a^T = a$, we find that the mth term of Eq. (B.5) becomes

$$\begin{aligned} &\bar{y}^{2n} \left(\frac{\sigma}{D\bar{y}}\right)^{2m} \frac{1}{D^{2m}} \sum_{i_1 + \cdots + i_{m+1} = n-m} ab^T(i_{m+1} + 1) \cdots \\ &\quad \times ab^T(i_{m+1} + \cdots + i_2 + m) a \\ &\quad \times ab(i_{m+1} + \cdots + i_2 + m) a \cdots b(i_{m+1} + 1) a \end{aligned} \tag{B.6}$$

In Eq. (B.6) we do not consider the case with $i_k = 0$, since it gives only a small correction for large n. Using the independence of $b(k)$ and the fact that $\overline{b(k)_{ij}} = 0$ $\overline{(b(k)_{ij})^2} = 1$, we calculate the average of each term of

Eq. (B.6), and noting that each term $\mathrm{b}^T(i_k)\mathrm{a}\cdots\mathrm{ab}(i_k)$ gives a contribution cD^2, where $c=2$ if the matrix b is symmetric and $c=1$ otherwise, we get (for large n)

$$c^m \bar{y}^{2n}\left(\frac{\sigma}{D\bar{y}}\right)^{2m} \sum_{i_1+\cdots+i_{m+1}=n-m} 1 \simeq c^m \bar{y}^{2n}\left(\frac{\sigma}{D\bar{y}}\right)^{2m} \frac{n^m}{m!} \tag{B.7}$$

From Eqs. (B.2), (B.6), and (B.7), we obtain

$$L(2)=2\ln \bar{y}+c\left(\frac{\sigma}{D\bar{y}}\right)^2 \tag{B.8}$$

Therefore, assuming the validity of Eq. (2.14), Eq. (3.8) follows from Eq. (B.8) upon noting that $L(1)=\ln \bar{y}$.

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